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An Abel–Tauber Theorem for Partitions, II

J. L. GELUK*

Department of Mathematical Sciences,
King Fahd University of Petroleum & Minerals,
Dhahran, Saudi Arabia

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Suppose $\Lambda = \{\lambda_1, \lambda_2, ...\}$ is a given set of real numbers such that $0 < \lambda_1 < \lambda_2 < \cdots$. Let $n(u) = \sum_{\lambda_k \leqslant u} 1$ and $P(u)$ the number of solutions of $n_1\lambda_1 + n_2\lambda_2 + \cdots \leqslant u$ in integers $n_i \geqslant 0$. We prove a relation between $n(u)$ and $\log P(u)$ for the case where n satisfies the condition

$$\lim_{x \to \infty} \overline{\lim_{t \to \infty}}\, n(tx)/n(t) < \infty.$$

INTRODUCTION

Suppose the sequence $\{\lambda_n\}$ with $0 < \lambda_1 < \lambda_2 < \cdots$ is given. We define the function n by

$$n(u) = \sum_{\lambda_k \leqslant u} 1 \tag{1}$$

and suppose that for all $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that

$$n(u) < c(\varepsilon)\, e^{\varepsilon u}. \tag{2}$$

Then $\prod_{r=1}^{\infty} (1 - e^{-\lambda_r s})^{-1}$ converges for $s > 0$. Consider the generating Dirichlet series defined by

$$\prod_{r=1}^{\infty} (1 - e^{-\lambda_r s})^{-1} = \sum_{n=0}^{\infty} p(v_n)\, e^{-s v_n}, \qquad s > 0, \tag{3}$$

* Present address: Erasmus University, Econometric Institute, Burg. Oudlaan 50, 3000 DR Rotterdam, The Netherlands.

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where $p(v_n)$ denotes the number of solutions of $\sum_j m_j \lambda_j = v_n$, where the summation is taken over non-negative integers and is finite. If we define

$$P(u) = \sum_{v_i \leqslant u} p(v_i), \tag{4}$$

then the right-hand side in (3) equals $\hat{P}(s)$, $\hat{P}$ denoting the Laplace–Stieltjes transform of P.

Generalizing earlier results of Hardy and Ramanujan [11], Knopp [14], Erdös [4], and Bingham [2], Kohlbecker [15] proved that the function n defined by (1) is regularly varying with exponent α ($\alpha > 0$), i.e., $\lim_{t \to \infty} n(tx)/n(t) = x^{\alpha}$ for $x > 0$, if and only if $\log P$ is regularly varying with exponent $\alpha/(\alpha + 1)$.

The special case $\alpha = 0$ is considered by Parameswaran [17]. The main result in his paper is that under suitable conditions we have

$$n(u) \sim L(u) \qquad (u \to \infty), \tag{5}$$

where L is slowly varying (i.e., $L(tx) \sim L(t)$ $(t \to \infty)$ for $x > 0$) if and only if $\log P(u) \sim [M^*(u)]^{-1}$ $(u \to \infty)$, where M^* is slowly varying. Moreover, if M is defined by $M(x) = \int_a^x L(s)/s\, ds$, then M and M^* form a pair of conjugate slowly varying functions in the sense of de Bruijn [3].

The last result can be improved in the sense that if n is slowly varying, then

$$\log P(y) = \int_0^x n(s)/s\, ds + n(x) + o(n(x)) \qquad (x \to \infty), \tag{6}$$

where $y \sim xn(x)$ $(x \to \infty)$ and a corresponding converse result (see [7]).

In this paper we obtain an analogue of (6), where $o(n(x))$ is replaced by $O(n(x))$. This result is obtained under the assumption

$$\lim_{x \to \infty} \varlimsup_{t \to \infty} n(tx)/n(t) < \infty. \tag{7}$$

Observe that since n is non-decreasing the limit as $x \to \infty$ exists (possibly infinite). The class of functions n satisfying $\varlimsup_{x \to \infty} \varlimsup_{t \to \infty} n(tx)/n(t) < \infty$ (without the assumption of monotonicity) was studied in a recent paper by Drasin and Seneta [4]. They prove that the above class consists of the functions n satisfying $n(u) \underset{\frown}{\smile} L(u)$ $(u \to \infty)$, where the function L is slowly varying.

We also give a converse result in which the asymptotic behaviour of n is obtained from the behaviour of $\log P$.

Condition (7) above is weaker than (5), but stronger than the assumption

$$\varlimsup_{t \to \infty} n(tx)/n(t) < \infty \qquad \text{for} \quad x > 1, \tag{8}$$

which is used in a paper by Schwarz [18, Satz 2]. Compared with Satz 2 in [18] our main result is restricted to a smaller class of functions n, but gives the behaviour of $\log P$ in terms of n and not in terms of $\log \hat{P}$.

The method of proof is similar to the one given in Geluk [7].

Results

In order to formulate the results, we first need three definitions.

Definition 1. A function $s\colon \mathbb{R}^+ := (0, \infty) \to \mathbb{R}^+$ belongs to the class Λ if s is decreasing,

$$\lim_{t \to \infty} ts(t) = \infty, \tag{9}$$

$$\varliminf_{t \to \infty} s(tx)/s(t) > 0 \qquad \text{for} \quad x > 1. \tag{10}$$

There exist M, $x_0 > 0$ such that

$$\varlimsup_{t \to \infty} s(tx)/s(t) < M/x \qquad \text{for all} \quad x \geqslant x_0. \tag{11}$$

Observe that since s is decreasing, (11) implies that $s(x) \to 0$ as $x \to \infty$.

Definition 2. If the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive and satisfies

$$\varlimsup_{t \to \infty} f(tx)/f(t) < \infty \qquad \text{for all} \quad x > 0, \tag{12}$$

then f is said to be O-regularly varying.

Definition 3. If f is O-regularly varying, then we define the upper and lower Matuszewska index by

$$\bar{\imath}(f) = \lim_{x \to \infty} \frac{\log \varlimsup_{t \to \infty} f(tx)/f(t)}{\log x} \tag{13}$$

and

$$i(f) = \lim_{x \to \infty} \frac{\log \underline{\lim}_{t \to \infty} f(tx)/f(t)}{\log x}, \tag{14}$$

respectively.

Remark. It follows from (11) that for $s \in \Lambda$ we have $\bar{\imath}(s) \leqslant -1$. Combination with (9) shows that $\bar{\imath}(s) = -1$ for $s \in \Lambda$. (This follows since if f satisfies $-\infty < i(f) \leqslant \bar{\imath}(f) < 0$, then $f(t) \to 0$ $(t \to \infty)$.)

MAIN THEOREM. *Suppose the function n as defined in* (1) *satisfies*

$$\lim_{x \to \infty} n(x) = \infty. \tag{15}$$

Then

$$\lim_{x \to \infty} \overline{\lim_{t \to \infty}}\, n(tx)/n(t) < \infty \tag{16}$$

implies

$$\log P(x) = \int_0^x s(t)\, dt + O(xs(x)) \qquad (x \to \infty), \tag{17}$$

where P is as defined in (4). *In this case the function s is related to n by $s(t) := \inf\{u; u^{-1}n(u^{-1}) \leqslant t\}$, $t > 0$. Conversely, if P satisfies* (17) *with some function s satisfying $s \in \Lambda$ and*

$$\sup_{1 \leqslant y \leqslant x} y^{-1}s^{\leftarrow}(y^{-1}) = O(x^{-1}s^{\leftarrow}(x^{-1})) \qquad (x \to \infty), \tag{18}$$

where $s^{\leftarrow}$ denotes the inverse of s, then

$$\int_0^x n(t)/t\, dt = \int_{1/x}^{\infty} s^{\leftarrow}(t)\, dt + O(x^{-1}s^{\leftarrow}(x^{-1})) \qquad (x \to \infty). \tag{19}$$

Remark. The de Bruijn conjugate is also present in the above result. If f is locally bounded and $f(t) \asymp L(t)$ $(t \to \infty)$, where L is slowly varying, let $g(t) := tf(t)$, take the generalized inverse g^+ of g, and define the de Bruijn conjugate by $f^*(t) := t^{-1}g^+(t)$. In the first part of the theorem we have $n(u) = u^{-1}s^{\leftarrow}(u^{-1})$, hence, $n^*(x) = 1/(xs(x))$, and in the second part the conjugate of $f(x) = 1/(xs(x))$ is $f^*(x) = x^{-1}s^{\leftarrow}(x^{-1})$.

For the proof of the main result we need seven lemmas. Since some of the lemmas are of independent interest we specify conditions on the functions for each lemma separately.

LEMMA 1 (see [1]). *Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive. Then the following statements are equivalent*

(i) *f is O-regularly varying.*

(ii) *there exist $\alpha, \beta \in \mathbb{R}$, t_0, and $c > 1$ such that*

$$c^{-1}x^{\beta} \leqslant \frac{f(tx)}{f(t)} \leqslant cx^{\alpha} \qquad \text{for all} \quad x \geqslant 1,\ t \geqslant t_0. \tag{20}$$

(iii) $-\infty \leqslant \underline{i}(f) \leqslant \bar{\imath}(f) < \infty$.

Moreover in inequality (20) *we may choose any $\beta < \underline{i}(f)$ and $\alpha > \bar{\imath}(f)$.*

As a consequence if f is in addition locally bounded on $[0, \infty)$, then for arbitrary $\xi > 0$ and $\beta < i(f)$, there exist $c > 0$ and t_0 such that $f(tx)/f(t) \leqslant cx^{\beta}$ for $t \geqslant t_0$ and $0 < x \leqslant \xi$.

LEMMA 2. *Suppose $n\colon \mathbb{R}^+ \to \mathbb{R}$ and $n(x)/x$ is integrable on finite intervals of $(0, \infty)$. If $n(x) = O(l(x))$ $(x \to \infty)$ where l satisfies $-1 < \underline{i}(l) \leqslant \bar{\imath}(l) < \infty$, then the function $\tilde{n}$ defined by*

$$\tilde{n}(s) = s \int_0^{\infty} \frac{e^{-us}}{1 - e^{-us}}\, n(u)\, du, \qquad s > 0 \tag{21}$$

satisfies

$$\tilde{n}(1/x) = \int_0^x \frac{n(t)}{t}\, dt + O(l(x)) \qquad (x \to \infty). \tag{22}$$

Proof. Without loss of generality we may assume $|n(x)/l(x)| \leqslant c_0$ for $x > 0$ and l bounded on finite intervals of $\mathbb{R}^+$. Since

$$\begin{aligned} &\frac{\int_0^x (n(t)/t)\, dt - \tilde{n}(1/x)}{l(x)} \\ &\quad = \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1 - e^{-u}}\right) \frac{n(xu)}{l(xu)} \frac{l(xu)}{l(x)}\, du \\ &\quad\quad - \int_1^{\infty} \frac{e^{-u}}{1 - e^{-u}} \frac{n(xu)}{l(xu)} \frac{l(xu)}{l(x)}\, du =: I_1 - I_2, \end{aligned}$$

we have for $x > x_0$ by Lemma 1

$$|I_1| \leqslant c_0 \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}\right) cu^\alpha \, du < \infty, \qquad \text{where} \quad -1 < \alpha < \underline{i}(l),\ c > 1$$

and

$$|I_2| \leqslant c_0 \int_1^\infty \frac{e^{-u}}{1-e^{-u}} cu^\beta \, du < \infty, \qquad \text{where} \quad \beta > \bar{\imath}(l).$$

LEMMA 3. *Suppose the function f is bounded on finite intervals of $(0, \infty)$ and $0 \leqslant \underline{i}(f) \leqslant \bar{\imath}(f) < \infty$. Then the generalized inverse function f^+ defined by $f^+(x) := \inf\{y; f(y) \geqslant x\}$ satisfies*

$$\underline{i}(f^+) = \frac{1}{\bar{\imath}(f)} \qquad \textit{and} \qquad \bar{\imath}(f^+) = \frac{1}{\underline{i}(f)}.$$

Proof. Without loss of generality we may assume that f is increasing and continuous (see [9, Corollary 3.7e and g]). Observe that $\bar{\imath}(f) = \inf \alpha$ and $\underline{i}(f) = \sup \beta$ where the inf and sup are taken over the values of α, β for which there exist $t_0 > 0$, $x_0 > 1$ such that

$$x^\beta f(t) \leqslant f(tx) \leqslant x^\alpha f(t) \qquad \text{for} \quad t \geqslant t_0,\ x \geqslant x_0 \text{ (see [9, Thm. 3.2]).}$$

It follows that $f^{\leftarrow}(tx) \geqslant x^{1/\alpha} f^{\leftarrow}(t)$ for $x \geqslant x_0^\alpha$, $t \geqslant f(t_0)$ and a similar lower inequality, from which the statement of the lemma follows.

LEMMA 4. *Suppose $n\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and $n(t)/t$ is integrable on $(0, 1)$. Then*

$$\tilde{n}(x) = \hat{\Psi}_n(x) + \int_0^1 \frac{n(u)}{u} du + O(x) \qquad (x \to 0+), \tag{23}$$

where $\hat{\Psi}_n$ is the Laplace–Stieltjes transform of the function Ψ_n, defined by

$$\Psi_n(x) = \sum_{1 \leqslant m \leqslant x} n(x/m)/m. \tag{24}$$

Suppose $s \in \Lambda$. Then the following are equivalent:

$$\tilde{n}(x) = \int_x^\infty s^{\leftarrow}(u) \, du + O(xs^{\leftarrow}(x)), \qquad x \to 0+ \tag{25}$$

$$\Psi_n(1/x) = \int_x^\infty s^{\leftarrow}(u) \, du + O(xs^{\leftarrow}(x)), \qquad x \to 0+. \tag{26}$$

Proof. Since

$$\tilde{n}(x) - x\int_0^1 \frac{e^{-ux}}{1-e^{-ux}}\, n(u)\, du$$

$$= x\int_1^\infty \sum_{k=1}^\infty e^{-kux} n(u)\, du$$

$$= \int_0^\infty e^{-ux}\, d\Psi_n(u) = \hat{\Psi}_n(x)$$

and

$$\int_0^1 \frac{uxe^{-ux}}{1-e^{-ux}} \frac{n(u)}{u}\, du = \int_0^1 \frac{n(u)}{u}\, du + O(x) \qquad (x \to 0+),$$

the proof of the first part is finished. Hence (25) is true if and only if $\hat{\Psi}_n(x) = \int_x^\infty s^{\leftarrow}(u)\, du + O(xs^{\leftarrow}(x))$, $x \to 0+$.

Since we have $s \in \Lambda$, hence $-\infty < \underline{i}(s) \leqslant \bar{i}(s) = -1$ and the function $s^{\leftarrow}(x^{-1})$ is O-regularly varying (see [9, Cor. 3.7.g]). Together with the above expression for $\hat{\Psi}_n(x)$ this gives

$$\overline{\lim_{x \to \infty}} \frac{\hat{\Psi}_n(1/ax) - \hat{\Psi}_n(1/x)}{x^{-1}s^{\leftarrow}(x^{-1})} < \infty \qquad \text{for all} \quad a > 1.$$

Since $y(x) := 1/s(x)$ satisfies $1 = \underline{i}(y) \leqslant \bar{i}(y) < \infty$, application of Lemma 3 shows that the lower index of the function $x^{-1}s^{\leftarrow}(x^{-1})$ is greater than -1, while $\bar{i}(x^{-1}s^{\leftarrow}(x^{-1})) = 0$. Hence, since Ψ is non-decreasing we may apply Theorem 3 in [10] to find

$$\Psi_n(1/x) - \hat{\Psi}_n(x) = O(xs^{\leftarrow}(x)) \qquad (x \to 0+),$$

from which (26) follows by combination with the above formula for $\hat{\Psi}_n$.

LEMMA 5. *Suppose A: $\mathbb{R}^+ \to \mathbb{R}^+$ is eventually non-decreasing,*

$$a(x) := x^{-1}\int_1^x A(t)\, dt \tag{27}$$

and

$$\overline{\lim_{x \to \infty}} \frac{a(\lambda x) - a(x)}{l(x)} < \infty \qquad \textit{for all} \quad \lambda > 1, \tag{28}$$

where l satisfies $xl(x) \to \infty$ as $x \to \infty$. Then

$$\overline{\lim_{x \to \infty}} \frac{A(x) - x^{-1}\int_1^x A(t)\, dt}{l(x)} < \infty. \tag{29}$$

Proof. Note that w.l.o.g. we may change the interval of integration in (29) to $(0, x)$ since $xl(x) \to \infty$ $(x \to \infty)$. For convenience we suppose $A(t) = 0$ on $(0, 1)$. Note that

$$x^{-1} \int_0^x A(t)\, dt = \int_0^x t^{-2} B(t)\, dt, \qquad \text{where} \quad B(t) = \int_0^t s dA(s).$$

As a consequence, for $\lambda > 1$ we have

$$\begin{aligned}\frac{a(\lambda x) - a(x)}{l(x)} &= \frac{(\lambda x)^{-1} \int_0^{\lambda x} A(s)\, ds - x^{-1} \int_0^x A(s)\, ds}{l(x)} \\ &= \int_1^\lambda \frac{B(xs)}{xl(x)} \frac{ds}{s^2} \geqslant \frac{B(x)}{xl(x)} \left(1 - \frac{1}{\lambda}\right).\end{aligned}$$

Hence $\overline{\lim}_{x \to \infty} B(x)/xl(x) < \infty$, which implies (29) by partial integration.

LEMMA 6. *Suppose the function Ψ_n defined by* (24) *satisfies* (26), *where $s \in A$ satisfies* (18). *If $n(t)/t$ is integrable on* $(0, 1)$ *and if $n(x)$ is non-decreasing for $x > 0$, then*

$$n(x) = O(x^{-1} s^{\leftarrow}(x^{-1})) \qquad (x \to \infty) \tag{30}$$

and (19) *holds.*

Proof. Define $A(x) = n(x) + \int_1^x n(t)/t\, dt$, a as in (27) and $N(x) = \sum_{1 \leqslant k \leqslant x} \mu(k)/k$, where μ is the Möbius function. Partial integration in the above formula for A and Möbius inversion in (24) gives

$$n(x) = \sum_{k \leqslant x} \frac{\mu(k)}{k} \Psi_n\left(\frac{x}{k}\right) = \frac{1}{x} \int_1^x t dA(t). \tag{31}$$

By substitution we find

$$\int_1^x \Psi_n(x/u) N(u)/u\, du = \sum_{k \leqslant x} \frac{\mu(k)}{k} \int_1^{x/k} \Psi\left(\frac{x}{ku}\right) \frac{du}{u} = \int_1^x \frac{1}{u} n\left(\frac{x}{u}\right) du = a(x).$$

Hence we have with $l(x) := x^{-1} s^{\leftarrow}(x^{-1})$

$$\begin{aligned}\frac{a(\lambda x) - a(x)}{l(x)} &= \int_1^x \left(\frac{\Psi_n(\lambda x/u) - \Psi_n(x/u)}{l(x/u)} \frac{l(x/u)}{l(x)}\right) \frac{N(u)}{u}\, du \\ &\quad + \int_1^\lambda \frac{\Psi_n(\lambda/u) N(xu)}{ul(x)}\, du.\end{aligned} \tag{32}$$

Note that (26) implies

$$\varlimsup_{x \to \infty} \{\Psi_n(\lambda x) - \Psi_n(x)\}/l(x) < \infty \qquad \text{for} \quad \lambda > 1.$$

Combination with (18) shows that the expression between brackets on the right-hand side in (32) is bounded as $x \to \infty$ (uniformly for $u \in (1, x)$). Since $\int_1^\infty |N(u)|/u \, du < \infty$ (see Landau [16]), the first integral in (32) is bounded as $x \to \infty$. Moreover since $s \in \Lambda$ we have $l(x) \to \infty$ as $x \to \infty$. Hence (28) is satisfied (observe that $N(x) \to 0$ as $x \to \infty$). Since $xn(x)$ is non-decreasing, A is non-decreasing for $x > 1$. Lemma 5 finally gives $n(x) = O(l(x))$ $(x \to \infty)$. Application of Lemma 2 now gives

$$\tilde{n}(1/x) = \int_0^x \frac{n(u)}{u} \, du + O(l(x)) \qquad (x \to \infty).$$

Since $l(x) \to \infty$ $(x \to \infty)$ application of the first part of Lemma 5 gives

$$\hat{\Psi}_n(1/x) = \int_0^x \frac{n(u)}{u} \, du + O(l(x)) \qquad (x \to \infty).$$

In view of $n(x) = O(l(x))$ this implies that for any $a > 1$

$$\varlimsup_{x \to \infty} \frac{\hat{\Psi}_n(1/ax) - \hat{\Psi}_n(1/x)}{l(x)} = \varlimsup_{x \to \infty} \int_1^a \frac{n(xu)}{l(xu)} \frac{l(xu)}{l(x)} \frac{du}{u} + O(1) < \infty.$$

Since Ψ_n is non-decreasing we may apply Theorem 3 on p. 348 in [10] to find

$$\Psi_n(x) - \hat{\Psi}_n(1/x) = O(l(x)) \qquad (x \to \infty), \tag{33}$$

hence,

$$\Psi_n(x) = \int_1^x \frac{n(u)}{u} \, du + O(l(x)) \qquad (x \to \infty).$$

Combination with (26) then gives (19).

LEMMA 7 (see [8, Theorem 10]). *Suppose* $f: \mathbb{R}^+ \to \mathbb{R}$ *is locally integrable,* $\log u \int_0^\infty f(t) \, e^{-ut} \, dt < \infty$ *for* $u > 0$ *and* $s \in \Lambda$ *(see Definition* 1*). Then*

$$\log f(t) = \int_0^t s(x) \, dx + O(ts(t)) \qquad (t \to \infty) \tag{34}$$

implies

$$\log u \int_0^\infty f(t)\, e^{-ut}\, dt = \int_u^\infty s^{\leftarrow}(x)\, dx + O(us^{\leftarrow}(u)), \qquad u \to 0+, \quad (35)$$

where $s^{\leftarrow}$ *denotes the inverse function of* s. *Conversely, if* f *is non-decreasing and* (11) *holds for all* $x > 1$, *then* (35) *implies* (34).

Proof (*of the Main Theorem*). Since n is non-decreasing, (16) implies that $\underline{i}(n) = \bar{i}(n) = 0$. Application of Lemma 1 then shows that (2) is satisfied. Hence, by (3) and (21),

$$\begin{aligned} \log \hat{P}(s) &= \log \sum_{n=0}^{\infty} p(v_n)\, e^{-sv_n} \\ &= -\int_0^\infty \log(1 - e^{-us})\, dn(u) = \tilde{n}(s), \end{aligned} \qquad (36)$$

both sides being well-defined for $s > 0$. Since n is zero on a right-hand neighbourhood of 0, the assumption of Lemma 2 are satisfied and we have

$$\begin{aligned} \log u &\int_0^\infty e^{-ux} P(x)\, dx \\ &= \log \hat{P}(u) = \tilde{n}(u) \\ &= \int_u^\infty n(x^{-1})\, x^{-1}\, dx + O(n(1/u)) \\ &= \int_u^\infty s^{\leftarrow}(x)\, dx + O(us^{\leftarrow}(u)) \qquad (u \to 0+). \end{aligned}$$

Next we verify that the assumptions of Lemma 7 are satisfied.

Since n is non-decreasing and $n(t) \to \infty$ $(t \to \infty)$, it follows that $s^{\leftarrow}(t) = t^{-1} n(t^{-1})$ is decreasing and $s^{\leftarrow}(t) \to \infty$ $(t \to 0+)$, hence, (9).

Since $n(u^{-1}) = us^{\leftarrow}(u)$ is non-increasing, $us(u)$ is non-decreasing, hence, (10). By (16) there exists $c > 0$ such that $\overline{\lim}_{t\to\infty} n(tx)/n(t) < c$ for all $x > 0$, or, equivalently, $\overline{\lim}_{t\to\infty} x^{-1} s^{\leftarrow}(t^{-1}x^{-1})/s^{\leftarrow}(t^{-1}) < c$ for all $x > 0$. Hence for $x > 0$ arbitrary there exists $t_0 = t_0(x)$ such that $xs^{\leftarrow}(tx) \leqslant cs^{\leftarrow}(t)$ for $t \leqslant t_0(x)$, which is equivalent to $s(cu/x)/s(u) \leqslant x$, $u \geqslant u_0(x)$. The last inequality then gives (11), for all $x > 1$. Since P is non-decreasing, application of Lemma 7 finally gives (17).

Conversely, if P satisfies (17) with $s \in \Lambda$ satisfying $\sup_{1 \leqslant y \leqslant x} y^{-1} s^{\leftarrow}(y^{-1}) = O(x^{-1} s^{\leftarrow}(x^{-1}))$, we find

$$\log \hat{P}(u) = \int_u^\infty s^{\leftarrow}(x)\, dx + O(us^{\leftarrow}(u)) \qquad (u \to 0+)$$

by application of Lemma 7. Now we can apply Lemmas 4 and 6 in order to find (19).

Remarks. Lemma 6 can be considered as an O-version of earlier results of Ingham, Segal and Jukes (see [12, 19, 13]).

For example, Ingham proved that if f is positive and non-decreasing for $x \geqslant 1$, then $\sum_{n \leqslant x} f(x/n) = x \log x + cx + o(x)$ $(x \to \infty)$ implies $f(x) \sim x$ $(x \to \infty)$ and $\int_1^\infty (f(x) - x)/x^2\, dx = c - \gamma$, where γ is Euler's constant. This result is generalized in [13, 19]. If we substitute $xn(x) = f(x)$ in Lemma 6 and $s^{\leftarrow}(x) = x^{-1}(\log x^{-1})^\alpha$ for $x \in (0, 1)$, $s^{\leftarrow}(x) = 0$ for $x \geqslant 1$, where $\alpha > 0$ is a constant, then $\sum_{n \leqslant x} f(x/n) = (\alpha + 1)^{-1} x(\log x)^{\alpha+1} + O(x(\log x)^\alpha)$ $(x \to \infty)$ implies $f(x) = O(x(\log x)^\alpha)$ and $\int_1^x f(t)/t^2\, dt = (\alpha+1)^{-1} (\log x)^{\alpha+1} + O(\log x)^\alpha$ $(x \to \infty)$.

Since the assumptions of the theorem are very general, our result gives estimates for partition functions which are weaker than the estimates obtained in special cases. The following example gives an idea about applicability of the theorem.

EXAMPLE. If $\log P(x) = (\log x)^\alpha + O(\log x)^{\alpha-1}$ $(x \to \infty)$, where $\alpha > 1$ is a constant, then

$$\int_1^x n(t)/t\, dt = (\log x)^\alpha + \alpha(\alpha - 1) \times (\log \log x)(\log x)^{\alpha-1} + O(\log x)^{\alpha-1}$$

$(x \to \infty)$. Conversely, if there exist constants $c_1, c_2, x_0 > 0$, $\alpha > 1$, such that $c_1(\log x)^{\alpha-1} \leqslant n(x) \leqslant c_2(\log x)^{\alpha-1}$ for $x \geqslant x_0$, then the asymptotic behaviour of P is given by (17) with $s(t) = \inf\{u; u^{-1}n(u^{-1}) \leqslant t\}$.

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